mechanics [1]. In the considered problems the necessary and sufficient condition of stability is of the form $\partial K_{II} / \partial l < 0$, i.e. by virtue of (5.8) and (5.10)

$$\frac{a+k}{G^+(-1)} > \frac{-9\eta_1 l}{4G^+(-2)} \qquad (5.12)$$

As an illustration of the derived solution we present below the dependence of the dimensionless critical height of the slope $K_{IIc} / (\delta H \sqrt{\pi l})$ on the dimensionless parameters $k / \delta H$ and l / H for a load-free slope for $\nu = 0{,}3$, $q_0 = q_1 = 0$, $\delta_0 = \delta_1 = 0$, $\beta = 120°$, $\alpha = 75°$ and $\rho = 30°$

$$\frac{K_{IIc}}{\delta H \sqrt{\pi l}} = 0.46 - 0.26 \frac{k}{\delta H} - 0.034 \frac{l}{H}$$

The obtained solution can be, evidently, used also for the experimental determination of parameter K_{IIc}, for instance, in experiments on uniaxial compression of specimens with an artificial boundary discontinuity along an inclined bonding plane (formula (5.9) for $\eta_1 = 0$ and a is equal to the right-hand part of equality (1.9)). The properties of the bond along the slip line and its continuation must simulate the properties of the filler in the tectonic crack and its interaction with the basic rock (quantities k and ρ of the adhesive must in any case be equal to the related minimum values of k and ρ that are characteristic for the pairs filler-filler and filler-rock in the limit and the sliding states). The practical difficulties of simulating the structure of the "head" slip line are, evidently, not smaller than in the case of crack of normal cleavage.

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SOME ELASTIC-PLASTIC PROBLEMS FOR A PLANE WEAKENED BY A PERIODIC SYSTEM OF CIRCULAR HOLES

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Elastic-plastic problems for a plane weakened by an infinite series of circular holes are considered. It is assumed that the stress level and the spacing between the holes are such that the circular holes are entirely enclosed by the appropriate

plastic zone but, at the same time, adjacent plastic domains do not merge.

The elastic-plastic problem for a plane with one hole was formulated and solved in [1]. A number of papers [2 — 4] has been devoted to periodic elastic-plastic problems. Elastic-plastic problems for a plane weakened by a doubly-periodic system of circular holes have been examined in [5, 6].

In contrast to [2 — 4], in which the small parameter method was used, another method is applied to solve the periodic elastic-plastic problems, which generally permits obtaining the solution for any relative dimensions of the domain .

Let there be a plane with circular holes having a radius R $(R < 1)$ and centers at the points

$$P_m = m\omega \quad (m = 0, \pm 1, \pm 2, \ldots), \quad \omega = 2$$

By L_m we denote the contour of a hole with center at the point P_m, by Γ_m the corresponding elastic-plastic boundary, and by D_z the exterior of the contours Γ_m.

The boundary conditions on the contour of the hole L_m are

$$\sigma_r = -p, \quad \tau_{r\theta} = 0 \tag{1}$$

We assume that the stress field in the plastic zone is

$$\begin{aligned} \sigma_r &= A / r^2 + B(1 + 2\ln r) + 2C \\ \sigma_\theta &= -A / r^2 + B(3 - 2\ln r) + 2C, \quad \tau_{r\theta} = 0 \end{aligned} \tag{2}$$

where A, B and C are certain constants. The axisymmetric stress field (2) satisfying the equilubrium equations is characterized by the fact that it permits compliance with certain plasticity conditions (see below) by an appropriate selection of the constants and by taking account of the plastic inhomogeneity, i.e. the dependence of the yield point on the coordinate r and on the principal stresses σ_θ and σ_r. At the same time, the method used for such a stress field, which is a combination of the method of solving the periodic elastic problem and the method proposed in [1] to solve elasticity and plasticity theory problems with an unknown boundary for a single hole, permits the effective solution of the elastic-plastic problem. The stresses in the elastic domain are determined by the Kolosov-Muskhelishvili formulas [7]

$$\begin{aligned} \sigma_r + \sigma_\theta &= 4\mathrm{Re}\,\Phi(z) \\ \sigma_\theta - \sigma_r + 2i\tau_{r\theta} &= 2[\bar{z}\Phi'(z) + \Psi(z)]e^{2i\theta} \end{aligned} \tag{3}$$

All the stresses are continous on the unknown contour Γ_m separating the elastic and plastic domains. Using (2) and (3), we obtain boundary conditions on the contour Γ_m

$$\begin{aligned} \mathrm{Re}\,\Phi(z) &= {}^1/_2 B \ln z\bar{z} + B + C \\ \bar{z}\Phi'(z) + \Psi(z) &= B\frac{\bar{z}}{z} - \frac{A}{z^2} \end{aligned} \tag{4}$$

Let us go over the parametric ζ-plane by using the transformation $z = \omega(\zeta)$. The analytic function $z = \omega(\zeta)$ maps the domain D_z conformally onto the domain D_ζ in the ζ-plane, which is the exterior of circles l_m of radius λ and centers at the points P_m.

To determine the three analytic functions $\varphi(\zeta) = \Phi[\omega(\zeta)]$, $\psi(\zeta) = \Psi[\omega(\zeta)]$ and $\omega(\zeta)$ we obtain the nonlinear boundary value problem on l_m

$$\mathrm{Re}\,\varphi(\zeta) = B + C + {}^1/_2 B \ln \omega(\zeta)\overline{\omega(\zeta)} \tag{5}$$

$$\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\varphi'(\zeta)+\psi(\zeta)=B\frac{\overline{\omega(\zeta)}}{\omega(\zeta)}-\frac{A}{[\omega(\zeta)]^2} \tag{6}$$

Solving the Dirichlet problem (5), we find that in the domain D_ζ

$$\varphi(\zeta)=B+C+B\ln\omega(\zeta)-B\ln\frac{\zeta}{\lambda} \tag{7}$$

Taking account of (7), the boundary condition (6) can be converted to

$$\omega'(\zeta)\,\omega^2(\zeta)\,\psi(\zeta)=B\frac{\overline{\omega(\zeta)}}{\zeta}\omega^2(\zeta)-A\omega'(\zeta) \tag{8}$$

We seek the required functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$ as the series

$$\varphi(\zeta)=\alpha_0+\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k)}(\zeta)}{(2k+1)!} \tag{9}$$

$$\psi(\zeta)=\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}\,\rho^{(2k)}(\zeta)}{(2k+1)!}-\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}s^{(2k+1)}(\zeta)}{(2k+1)!}$$

$$\omega(\zeta)=\zeta+\sum_{k=0}^{\infty}A_{2k+2}\frac{\lambda^{2k+2}\rho^{(2k-1)}(\zeta)}{(2k+1)!}$$

Here

$$\rho(\zeta)=\left(\frac{\pi}{\omega}\right)^2\sin^{-2}\left(\frac{\pi}{\omega}\zeta\right)-\frac{1}{3}\left(\frac{\pi}{\omega}\right)^2$$

$$S(\zeta)=\sum_m{}'\left[\frac{P_m}{(\zeta-P_m)^2}-\frac{2\zeta}{P_m^2}-\frac{1}{P_m}\right]$$

The prime on the summation sign means that the subscript $m=0$ is excluded in the summation.

Let us present the dependences which the coefficients of the expressions (9) should satisfy. We find from the conditions of symmetry relative to the coordinate axes that

$$\operatorname{Im}\alpha_{2k+2}=\operatorname{Im}\beta_{2k+2}=\operatorname{Im}A_{2k+2}=0,\quad k=0,1,2,\ldots \tag{10}$$

It is easy to see that the relationships (9),(10) define a class of symmetric problems with a periodic stress distribution.

From the condition that the principal vector of the forces acting on an arc connecting two congruent points in D_ζ is zero there follows that

$$\alpha_0={}^1/_{24}\,\pi^2\beta_2\lambda^2$$

By virtue of the periodicity conditions, the system of boundary conditions (8) on l_m ($m=0,\pm1,\pm2,\ldots$) is replaced by one functional equation on the contour l_0, say.

To form the equations in the remaining coefficients of the expressions (9) for the functions $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$, let us expand these functions in Laurent series in the neighborhood of the point $\zeta=0$

$$\varphi(\zeta)=\alpha_0+\sum_{k=0}^{\infty}\alpha_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}}+\sum_{k=0}^{\infty}\alpha_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}\zeta^{2j} \tag{11}$$

$$\psi(\zeta)=\sum_{k=0}^{\infty}\beta_{2k+2}\frac{\lambda^{2k+2}}{\zeta^{2k+2}}+\sum_{k=0}^{\infty}\beta_{2k+2}\lambda^{2k+2}\sum_{j=0}^{\infty}r_{j,k}\zeta^{2j}-$$

$$\sum_{k=0}^{\infty} (2k+2)\,\alpha_{2k+2}\,\lambda^{2k+2} \sum_{j=0}^{\infty} (2j+2k+2)\, r_{j,k}\zeta^{2j}$$

$$\omega(\zeta) = \zeta - \sum_{k=0}^{\infty} A_{2k+2} \frac{\lambda^{2k+2}}{(2k+1)\,\zeta^{2k+1}} + \sum_{k=0}^{\infty} A_{2k+2}\lambda^{2k+2} \sum_{j=0}^{\infty} \frac{r_{j,k}\zeta^{2j+1}}{2j+1}$$

Here

$$r_{j,k} = \frac{(2j+2k+1)!\, g_{j+k+1}}{(2j)!\,(2k+1)!\, 2^{2j+2k+2}}, \quad g_{j+k+1} = 2\sum_{m=1}^{\infty} \frac{1}{m^{2j+2k+2}}$$

Substituting their expansions (11) in place of $\varphi(\zeta)$, $\psi(\zeta)$ and $\omega(\zeta)$ in the boundary conditions (5) and (8) on the contour l_0 ($\zeta = \lambda e^{i\theta}$) and comparing coefficients of $e^{2ik\theta}$ ($k = 0, \pm 1, \pm 2, \ldots$), we obtain an infinite system of nonlinear algebraic equations in α_{2k}, β_{2k}, A_{2k} (condition (5) was preliminarily differentiated with respect to θ). The first approximation equations are presented below

$$XD + YD_2 + ZD_1 = B(Da - A_2D_1 + a_1D_2) - \frac{Aa}{\lambda^2}$$

$$XD_1 + YD = B(Da_1 + aD_1) - \frac{AA_2}{\lambda^2}$$

$$XD_2 + ZD = B(D_2a - A_2D) - AA_2\lambda^2 r_{1,0}$$

$$2\alpha_2(1 + \lambda^4 r_{1,0})\, d = Bd_1$$

$$X = a\beta_2 + A_2\beta_4\lambda^4 r_{1,0} + A_2\gamma_0, \qquad Y = a\beta_4 + A_2\beta_2$$

$$Z = a\gamma_0 + A_2\gamma_1 + A_2\beta_2\lambda^4 r_{1,0}, \qquad a = 1 + A_2\lambda^2 r_{0,0}$$

$$D = a^2 - {}^2/_3\, A_2{}^2\lambda^4 r_{0,1}, \qquad D_1 = -2aA_2$$

$$D_2 = {}^2/_3\, A_2 a\lambda^4 r_{0,1}, \qquad a_1 = {}^1/_3\, A_2\lambda^4 r_{1,0}$$

$$d = a^2 + A_2{}^2(1 + {}^1/_9\lambda^8 r_{1,0}{}^2), \qquad d_1 = -2aA_2(1 - 1/3\lambda^4 r_{1,0})$$

$$\gamma_j = \beta_2 r_{j,0}\lambda^{2j+2} + \beta_4 r_{j,1}{}^{2j+4} - 2(2j+2)\alpha_2\lambda^{2j+2} r_{j,0} \quad (j = 0,1).$$

To obtain the relationships relating the parameter λ to the applied load p, we substitute the first and third formulas in (9) into the boundary condition (5), then we multiply the expression obtained by $1/2\pi i\zeta$ and integrate over the circular contour l_0. We consequently obtain [7]

$$\alpha_0 + \sum_{k=0}^{\infty} \alpha_{2k+2}\,\lambda^{2k+2}\, r_{0,k} = B + C + B\ln\lambda \left[1 + \sum_{k=0}^{\infty} A_{2k+2}\,\lambda^{2k+2}\, r_{0,k}\right]$$

The boundary conditions (1) on the contour of the hole L_m and the flow conditions define the quantities A, B and C.

Let us consider some particular cases.

Tresca-Saint Venant or Huber-Mises plasticity condition. Let the relationship $|\sigma_\theta - \sigma_r| = 2k$ be satisfied in the plastic zone (k is the plasticity constant). In this case we have according to (1) and (2)

$$A = 0, \quad B = \varepsilon k, \quad 2C = -p - \varepsilon k(1 + 2\ln R)$$

Here $\varepsilon = \pm 1$ is selected from physical considerations. The results of a computation in the first two approximations are given in Table 1. In Fig. 1 the solid lines are dependences of the parameter λ on the magnitude of the applied load p/k for values R =0.5, 0.4, 0.3, 0.2, 0.1 of the hole radius (curves *1*—*5*).

Inhomogeneously-plastic material. Now, let the plasticity condition be [8]

$$\sigma_\theta - \sigma_r = 2\left[k_0 + k_1\left(\frac{R}{r}\right)^2\right]$$

Here k_0 and k_1 are material constants. This plasticity condition can be considered as the ordinary Tresca–Saint-Venant condition with a yield point dependent on the radius. In this case we have according to (1) and (2)

$$A = -k_1R^2, \quad B = k_0, \quad 2C = k_1 - k_0 - p - 2k_0 \ln R$$

The results of a computation in the second approximation are given in Table 2 for values of the inhomogeneity parameter $k_1R^2 / k_0 = 0{,}09$.

Dependences of the parameter λ on the quantity p/k_0 for $R = 0.3$ are represented by curves 6, 7 in Fig. 1 for values of the inhomogeneity parameter k_1/k_0 equal to 1 and -0.5, respectively.

Exponential flow condition. Let the plasticity condition be

$$\sigma_r - \sigma_\theta = 2k\left[1 - \exp\left(-\frac{\sigma_0}{k} + \frac{\sigma_r + \sigma_\theta}{2k}\right)\right]$$

Here $k > 0$ and $\sigma_0 > 0$ are material constants with the dimensionality of a stress.

This flow condition describes the limit state of some rock [9]. In this case, we have according to (1) and (2)

$$A = -ke^{-2}t^{-2}R^2, \quad B = -k, \quad C = {}^1/_2\sigma_0 - k \ln tR^{-1}$$

where t is a constant which is the root of the equation

$$k^{-1}(\sigma_0 + p) - 1 = e^{-2}t^{-2} + 2 \ln t \quad (t > e^{-1})$$

More general flow conditions. Let the following flow conditions hold in the plastic zone [10]:

$$\sigma_r - \sigma_\theta = 2\left[k + \frac{b}{r^2} - \frac{k}{r^\gamma}\exp\left(-\frac{\sigma_\theta}{k} + \frac{\sigma_r + \sigma_\theta}{2^{\gamma+1}k}\right)\right] \quad (\gamma = 0, 1)$$

According to (1) and (2) in this case, the constants A, B and C have the values

$$A = b - ke^{\gamma-2}t^{-2}R^2, \quad B = -k, \quad C = 2^{\gamma-1}[\sigma_0 + k \ln (R^2t^{-2})]$$

Here t is a constant which is the root of the equation

$$k^{-1}\left(\sigma_0 + p + \frac{b}{R^2}\right) - 1 = e^{-2}t^{-2} + 2\ln t \quad (t > e^{-1}) \quad \text{for } \gamma = 0$$

$$k^{-1}\left(2\sigma_0 + p + \frac{b}{R^2}\right) - 1 + 2\ln R = 4\ln t + e^{-1}t^{-2} \quad (t > e^{-1}) \quad \text{for } \gamma = 1$$

Setting $\zeta = \lambda e^{i\theta}$, in the last relationship in (11), we obtain the equation of the elastic-plastic boundary $r = |\omega(\lambda e^{i\theta})| = f(\theta)$. In a first approximation $r^2 = \lambda^2(d + d_1 \cos 2\theta)$, where

$$r_{\max} = \lambda\left[1 + A_2\left(-1 + \lambda^2 \sum_{j=0}^{\infty} \frac{r_{j,0}}{2j+1}\lambda^{2j}\right)\right] \quad (12)$$

$$r_{\min} = \lambda\left[1 + A_2\left(1 + \lambda^2 \sum_{j=0}^{\infty} \frac{(-1)^j r_{j,0}}{2j+1}\lambda^{2j}\right)\right] \quad (13)$$

The elastic-plastic boundary (one-fourth of the contour) is represented in Fig. 2 for the Tresca–Saint-Venant flow condition for the case $R = 0.3$, $p = 2.12\,k$ ($\lambda = 0.7$,

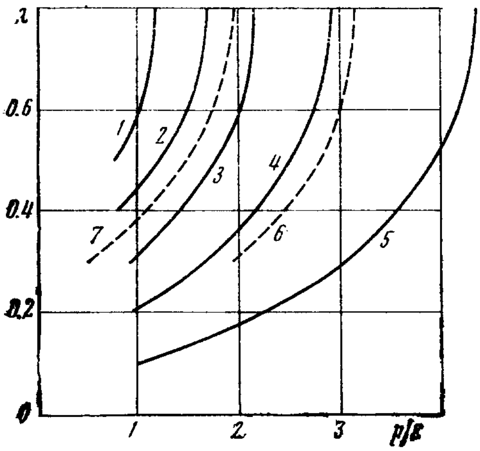

Fig. 1

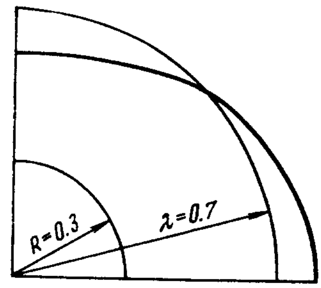

Fig. 2

Table 1

λ	0.2	0.4	0.6	0.8
		First approximation		
β_2/k	1.00060	1.00769	1.02267	1.05520
β_4/k	0.02908	0.08726	0.14352	0.20279
A_2	—0.02904	—0.08592	—0.13708	—0.18362
α_2/k	0.02902	0.08502	0.13037	0.15692
		Second approximation		
β_2/k	1.00061	1.00758	1.02066	1.04468
β_4/k	0.02911	0.08783	0.14264	0.19617
β_6/k	0.00238	0.01821	0.02467	0.02734
A_2	—0.02903	—0.08549	—0.13500	—0.17599
A_4	—0.00155	—0.01048	—0.00477	0.00573
α_2/k	0.29010	0.08460	0.12846	0.15130
α_4/k	0.00010	0.00009	—0.00605	—0.01312

Table 2

λ	0.2	0.4	0.6	0.8
β_2/k_0	3.27270	1.60176	1.31016	1.23598
β_4/k_0	—0.33327	—0.01791	0.08694	0.16020
β_6/k_0	0.10828	0.08469	0.08332	0.08842
A_2	—0.09458	—0.13470	—0.17103	—0.20369
A_4	—0.00504	—0.01644	—0.00635	0.00494
α_2/k_0	0.09394	0.13269	0.16259	0.17564
α_4/k_0	—0.00273	—0.00317	—0.01059	—0.01715

$r_{\max} = 0{,}81,\ r_{\min} = 0.58$).

The least load for which the hole contour is enclosed entirely by the plastic zone is determined from the condition $r_{\min} \geqslant R$. For $r_{\max} \leqslant 1$ the relationship (13) permits finding the largest load for which the plastic zones touch one another.

Up to now the mean stresses in the plane have been taken equal to zero. Let the mean stresses (tension or compression at infinity)

$$\sigma_x = \sigma_x^\infty, \quad \sigma_y = \sigma_y^\infty, \quad \tau_{xy} = 0$$

hold in the plane. In this case the complex potentials are sought in the form

$$\varphi_*(\zeta) = \frac{\sigma_x^\infty + \sigma_y^\infty}{4} + \varphi(\zeta), \quad \psi_*(\zeta) = \frac{\sigma_y^\infty - \sigma_x^\infty}{2} + \psi(\zeta)$$

where $\varphi(\zeta)$ and $\psi(\zeta)$ are determined by the first two relationships (9).

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